A Quasi-Hopf algebra interpretation of quantum 3-j and 6-j symbols and difference equations.

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Abstract

We consider the universal solution of the Gervais-Neveu-Felder equation in the $\mathcal{U}_q(sl_2)$ case. We show that it has a quasi-Hopf algebra interpretation. We also recall its relation to quantum 3-j and 6-j symbols. Finally, we use this solution to build a q-deformation of the trigonometric Lamé equation.

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1 Introduction

The Gervais-Neveu-Felder equation is a deformation of the standard Yang-Baxter equation. In the sl_2 case, it reads

$$R_{12}(x)R_{13}(xq^{H_2})R_{23}(x) = R_{23}(xq^{H_1})R_{13}(x)R_{12}(xq^{H_3}) \tag{1}$$

Here, H denotes a Cartan generator in sl_2 (or rather $\mathcal{U}_q(sl_2)$) and x is a parameter not to be confused with the spectral parameter (absent in the sl_2 case).

This equation appeared independently in several contexts. It was first discovered by J.L. Gervais and A. Neveu in their studies on Liouville theory [1]. It was rediscovered by G. Felder in his approach to the quantization of the Knizhnik-Zamolodchikov-Bernard equation [2]. Finally, it was shown to play an important role in the quantization of the Calogero-Moser models in the R-matrix formalism [3]. For all these reasons, we believe that this equation deserves much attention.

In this note, we analyse the universal solution $R(x) \in \mathcal{U}_q(sl_2) \otimes \mathcal{U}_q(sl_2)$ of eq.(1) obtained in [4]. We show that it has a nice quasi-Hopf algebra interpretation. For completeness, we recall the connection of this solution with q-analogs of 3-j and 6-j symbols. Finally, we explain how it can be used to construct a q-difference analog of the trigonometric Lamé equation (Calogero model for 2 particules).

2 A summary of universal formulae

In this section, we recall the universal formulae obtained in [4] for the matrix $R_{12}(x) \in \mathcal{U}_q(sl_2) \otimes \mathcal{U}_q(sl_2)$. We denote by $H, E_\pm$ the generators of the quantum group $\mathcal{U}_q(sl_2)$

$$[H, E_\pm] = \pm 2E_\pm, \qquad [E_+, E_-] = \frac{q^H - q^{-H}}{q - q^{-1}}$$

The coproduct is defined as

$$\Delta(H) = H \otimes id + id \otimes H, \qquad \Delta(E_\pm) = E_\pm \otimes q^{\frac{1}{2}H} + q^{-\frac{1}{2}H} \otimes E_\pm$$

We have $R^D_{12}\Delta(a) = \Delta'(a)R^D_{12}$ for any $a \in U_q(sl_2)$ where Δ' is the opposite comultiplication and R^D_{12} Drinfeld's universal R-matrix :

$$R^D_{12} = q^{\frac{1}{2}H\otimes H} \sum_{i=0}^{\infty} (q - q^{-1})^i \, \frac{q^{-\frac{i(i+1)}{2}}}{[i]!} \, q^{\frac{i}{2}H} E^i_+ \otimes q^{-\frac{i}{2}H} E^i_-$$

As usual, q-numbers are defined as $[i] = (q^i - q^{-i})/(q - q^{-1})$. Let us now define

$$R_{12}(x) = F^{-1}_{21}(x) \; R^D_{12} \; F_{12}(x) \tag{2}$$

with

$$\begin{aligned} F_{12}(x) &= \sum_{k=0}^{\infty} (q - q^{-1})^k \frac{(-1)^k}{[k]!} \frac{x^k}{\prod_{\nu=k}^{2k-1}(xq^\nu q^{H_2} - x^{-1}q^{-\nu}q^{-H_2})} q^{\frac{k}{2}(H_1+H_2)} E^k_+ \otimes E^k_- \\ F^{-1}_{12}(x) &= \sum_{k=0}^{\infty} (q - q^{-1})^k \frac{1}{[k]!} \frac{x^k}{\prod_{\nu=1}^{k}(xq^\nu q^{H_2} - x^{-1}q^{-\nu}q^{-H_2})} q^{\frac{k}{2}(H_1+H_2)} E^k_+ \otimes E^k_- \end{aligned} \tag{3}$$

It follows from the construction of [4] that $R_{12}(x)$ is a solution of eq.(1).

One can check that $F_{12}(x)$ satisfies the following "shifted cocycle" condition

$$[(id \otimes \Delta)F] \cdot [id \otimes F] = [(\Delta \otimes id)F] \cdot \Big[F(xq^{H_3}) \otimes id\Big] \tag{4}$$

This relation is proved using standard q-binomial identities. It turns out that $F_{12}(x)$ is actually a "shifted coboundary"

$$F_{12}(x) = \Delta M(x) \ \Big[id \otimes M(x)\Big]^{-1} \Big[M(xq^{H_2}) \otimes id\Big]^{-1} \tag{5}$$

where the formula for the "boundary" reads

$$M(x) = \sum_{n,m=0}^{\infty} \frac{(-1)^m x^m q^{\frac{1}{2}n(n-1)+m(n-m)}}{[n]![m]! \prod_{\nu=1}^{n} (xq^{\nu} - x^{-1}q^{-\nu})} E_+^n E_-^m q^{\frac{1}{2}(n+m)H}$$

Equation (5) implies eq. (4).

3 Quasi-Hopf algebra interpretation

The previous construction possesses a natural quasi-Hopf interpretation. Indeed, since $R(x)$ is defined in eq.(2) by a twisting procedure in the sense of Drinfeld [5] it is canonically associated to a quasi-Hopf structure on $U_q(sl_2)$. We shall denote it as $U_{q;x}(sl_2)$. This quasi-Hopf algebra possesses very specific properties due to the "shifted cocycle" relation (4) satisfied by $F(x)$. Besides Drinfeld's construction, this gives another example of a quasi-Hopf algebra structure over $\mathcal{U}_q(sl_2)$.

Let us recall following ref.[5] that a quasi-Hopf algebra is specified by a quadruplet (A, Δ, R, Φ) where A is an assocative algebra, Δ is a (non-coassociative) comultiplication in A, $R \in A \otimes A$ and $\Phi \in A \otimes A \otimes A$ are such that :

$$\begin{aligned} R\Delta(a) &= \Delta'(a)R & (6) \\ (id \otimes \Delta)\Delta(a) \ \Phi &= \Phi \ (\Delta \otimes id)\Delta(a) & (7) \end{aligned}$$

for all $a \in A$. There also are extra compatibility relations between Δ, R and Φ which we shall mention when needed. We will consider quasitriangular quasi-Hopf algebra, i.e., R is assumed to verify the conditions

$$\begin{aligned} (\Delta \otimes id)R &= \Phi_{321} R_{13} \Phi_{132}^{-1} R_{23} \Phi_{123} \\ (id \otimes \Delta)R &= \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi_{123}^{-1} \end{aligned}$$

There exists a twisting procedure to construct quasi-Hopf algebras. Namely, if (A, Δ, R, Φ) is a quasitriangular quasi-Hopf algebra then a new quasitriangular quasi-Hopf algebra $(A, \widetilde{\Delta}, \widetilde{R}, \widetilde{\Phi})$ is defined by $\widetilde{\Delta}(a) = F_{12}^{-1}\Delta(a)F_{12}$, and

$$\begin{aligned} \widetilde{\Phi} &= F_{23}^{-1}[(id \otimes \Delta)(F^{-1})] \ \Phi \ [(\Delta \otimes id)(F)]F_{12} & (8) \\ \widetilde{R} &= F_{21}^{-1} R F_{12} & (9) \end{aligned}$$

with $F_{12} \in A \otimes A$.

In our case, we are twisting $U_q(sl_2) \equiv (U_q(sl_2), \Delta, R^D, id)$ by $F(x)$. So we have $\widetilde{R} = F_{21}^{-1}(x) R_{12}^D F_{12}(x) = R(x)$ as defined in eq.(2). We denote $\widetilde{\Delta}$ by Δ_x with :

$$\Delta_x(a) = F_{12}^{-1}(x)\Delta(a)F_{12}(x), \quad \forall a \in U_q(sl_2) \tag{10}$$

It is a simple check to verify that the "shifted cocycle" condition (4) implies that :

$$(id \otimes \Delta_x)\Delta_x(a) = (\Delta_{xq^{H_3}} \otimes id)\Delta_x(a) \tag{11}$$

In other words, the shift breaks the co-associativity. We denote $\widetilde{\Phi}$ by $\Phi(x)$. It possesses a simple expression in terms of $F(x)$:

$$\begin{aligned} \Phi(x) &= F_{23}^{-1}(x)[(id \otimes \Delta)(F^{-1}(x))][(\Delta \otimes id)(F(x))]F_{12}(x) \\ &= F_{12}^{-1}(xq^{H_3})\ F_{12}(x) \end{aligned} \tag{12}$$

where in the last equality we again used the "shifted cocycle" relation (4).

We can now write all the quasi-Hopf relations in $U_{q;x}(sl_2)$ in terms of $R(x)$ or $F(x)$. For example, the general relation (7) reduces to eq.(11). Also, thanks to the following property,

$$R_{12}(xq^{H_3}) = \Phi_{213}(x)R_{12}(x)\Phi_{123}^{-1}(x)$$

the quasi- Yang-Baxter equation,

$$\Phi_{321}^{-1}(x)R_{12}(x)\Phi_{312}(x)R_{13}(x)\Phi_{132}^{-1}R_{23}(x) = R_{23}(x)\Phi_{231}^{-1}(x)R_{13}(x)\Phi_{213}(x)R_{12}(x)\Phi_{123}^{-1}(x)$$

valid in any quasitriangular quasi-Hopf algebra reduces to the equation (1).

Similarly, the quasitriangular property of $U_{q;x}(sl_2)$ implies that

$$\begin{aligned} (\Delta_x \otimes id)R(x) &= R_{13}(xq^{H_2})R_{23}(x)F_{12}^{-1}(xq^{H_3})F_{12}(x) \\ (id \otimes \Delta_x)R(x) &= F_{23}^{-1}(x)F_{23}(xq^{H_1})R_{13}(x)R_{12}(xq^{H_3}) \end{aligned}$$

Notice that for $x = 0$, $F_{12}(x)|_{x=0} = 1$ and therefore $R(x)|_{x=0} = R^D$. In the limit $x = \infty$, $F_{12}^{-1}(x)|_{x=\infty} = q^{-H\otimes H/2}R_{12}^D$. Thus $R_{12}(x)|_{x=\infty} = q^{-H\otimes H/2}R_{21}^D q^{H\otimes H/2}$, and $\Delta_{x=\infty}(a) = q^{-H\otimes H/2}\Delta'(a)q^{H\otimes H/2}$ for all $a \in U_q(sl_2)$.

4 Relation to 3-j and 6-j symbols

We now give a list of formulae expressing the matrix elements of the various objects we have considered so far in terms of standard q-analogs of the 3-j and 6-j symbols. Let $\rho^{(j)}$ denote the spin j representation of $\mathcal{U}_q(sl_2)$. Then

$$\begin{aligned} \rho^{(j)}(H)|j,m\rangle &= 2m\ |j,m\rangle \\ \rho^{(j)}(E_\pm)|j,m\rangle &= \sqrt{[j \mp m][j \pm m + 1]}\ |j, m \pm 1\rangle \end{aligned}$$

The first step is to find the matrix elements of the matrix $M(x)$ in the spin-j representation. We get

$$\begin{aligned} \left[M^{(j_1)}(x)\right]_{\sigma_1\, m_1} &= (-1)^{\sigma_1 + m_1} \frac{\sqrt{[j_1+\sigma_1]![j_1-\sigma_1]![j_1+m_1]![j_1-m_1]!}}{\prod_{r=1}^{j_1+\sigma_1}(1 - x^2 q^{2r})} \cdot \\ &\cdot q^{\sigma_1(\sigma_1 - m_1)} x^{\sigma_1 - m_1} \sum_p \frac{q^{2p\,\sigma_1} x^{2p}}{[p]!\ [\sigma_1 - m_1 + p]! [j_1 - \sigma_1 - p]!\ [j_1 + m_1 - p]!} \end{aligned} \tag{13}$$

This formula agrees (up to normalizations) with the one found in [10].

This matrix $M(x)$ is known to perform the vertex-IRF transformation in conformal field theory [6, 7, 8, 9].

$$\xi^{(j_1)}_{m_1}(z) = \sum_{\sigma_1} \psi^{(j_1)}_{\sigma_1}(z) M^{(j_1)}_{\sigma_1 m_1}(x)$$

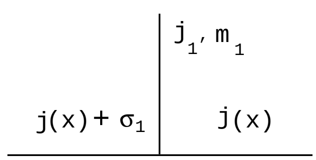
where the ψ's are IRF type operators and the ξ's are vertex type operators. The braiding relations of the ψ's are described by the matrix $R(x)$, while those of the ξ's are described by R^D. Thus, we expect the elements $M^{(j_1)}_{\sigma_1 m_1}(x)$ to be related to quantum 3-j symbols. The precise connexion was found in [11]. We have

$$\left[M^{(j_1)}(x)\right]_{\sigma_1\, m_1} \quad = \quad \frac{\mathcal{N}^{(j_1)}_{\psi}(x,\sigma_1)}{\mathcal{N}^{(j_1)}_{\xi}(m_1)} \lim_{m\to\infty} \begin{bmatrix} j_1 & j(x) & j(x)+\sigma_1 \\ m_1 & m & m+m_1 \end{bmatrix}_q \tag{14}$$

where we have defined $j(x)$ through the relation

$$x = q^{2j(x)+1} \tag{15}$$

Eq.(14) has to be understood as an analytic continuation in $j(x)$ of 3-j symbols [12]. We give a sketch of the proof in the Appendix. The factors $\mathcal{N}^{(j_1)}_{\xi}$ and $\mathcal{N}^{(j_1)}_{\psi}$ can be reabsorbed into the normalizations of the fields ψ and ξ respectively. Their expression is also given in the Appendix. We represent eq.(14) by a diagram

$$[M^{(j_1)}(x)]_{\sigma_1 m_1} = \tag{16}$$


From eq.(14), it is now possible to build the complete dictionary between the matrix elements of $F_{12}(x)$ and $R_{12}(x)$ and standard 3-j and 6-j symbols.

We start with

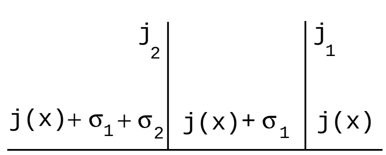
$$\langle j_1,\sigma_1|\langle j_2,\sigma_2|M^{(j_2)}_2(xq^{H_1})M^{(j_1)}_1(x)|j_1,m_1\rangle|j_2,m_2\rangle \quad = \quad M^{(j_2)}_{\sigma_2,m_2}(xq^{2\sigma_1})M^{(j_1)}_{\sigma_1,m_1}(x) =$$

$$\frac{\mathcal{N}^{(j_1)}_{\psi}(x,\sigma_1)\mathcal{N}^{(j_2)}_{\psi}(xq^{2\sigma_1},\sigma_2)}{\mathcal{N}^{(j_1)}_{\xi}(m_1)\mathcal{N}^{(j_2)}_{\xi}(m_2)} \lim_{m,m'\to\infty} \begin{bmatrix} j_2 & j(x)+\sigma_1 & j(x)+\sigma_1+\sigma_2 \\ m_2 & m & m+m_2 \end{bmatrix} \begin{bmatrix} j_1 & j(x) & j(x)+\sigma_1 \\ m_1 & m' & m'+m_1 \end{bmatrix}$$

Notice that we have used the fact that

$$xq^{2\sigma_1} = q^{2(j(x)+\sigma_1)+1}$$

Hence the shift in the Gervais-Neveu-Felder equation $x \to xq^H$ precisely corresponds to the shift of spins $j(x) \to j(x)+\sigma$. Thus we have the diagramatic correspondance

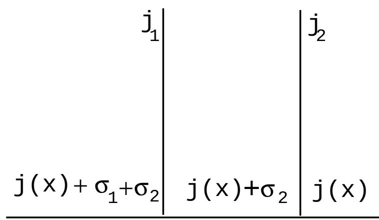
$$M^{(j_2)}_2(xq^{H_1})M^{(j_1)}_1(x) = \tag{17}$$


The matrix elements of $R_{12}(x)$ are computed from the formula

$$R_{12}(x)M_1(xq^{H_2})M_2(x) = M_2(xq^{H_1})M_1(x)R_{12}^D$$

or graphically

$$\sum_{\sigma_1\sigma_2} R(x)^{j_1 j_2}_{\sigma_1'\sigma_2',\sigma_1\sigma_2}$$



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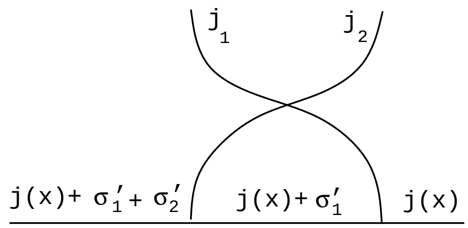

(18)

This is equivalent to the braiding relation and relates the matrix elements of $R(x)$ to 6-j symbols:

$$\begin{aligned}\langle j_1,\sigma_1'|\langle j_2,\sigma_2'|R_{12}(x)|j_1,\sigma_1\rangle|j_2,\sigma_2\rangle \;&=\; (-1)^{\sigma_1'-\sigma_1} q^{C(j(x))+C(j(x)+\sigma_1+\sigma_2)-C(j(x)+\sigma_1')-C(j(x)+\sigma_2)} \\ &\frac{\mathcal{N}_\psi^{(j_1)}(x,\sigma_1')\mathcal{N}_\psi^{(j_2)}(xq^{2\sigma_1'},\sigma_2')}{\mathcal{N}_\psi^{(j_1)}(xq^{2\sigma_2},\sigma_1)\mathcal{N}_\psi^{(j_2)}(x,\sigma_2)} \begin{Bmatrix} j_2 & j(x)+\sigma_1+\sigma_2 & j(x)+\sigma_1' \\ j_1 & j(x) & j(x)+\sigma_2 \end{Bmatrix}_q\end{aligned}$$

where $C(j) = j(j+1)$ and the last symbol represents a 6-j coefficient (see eq. 5.11 in [13]).

Finally, we give the formula for the matrix elements of $F_{12}(x)$ in terms of 3-j and 6-j symbols. We start from the formula

$$F_{12}(x)M_1(xq^{H_2})M_2(x) = \Delta M(x)$$

From the definition of the coproduct, we have

$$[\Delta M^{j_1 j_2}(x)]_{\sigma_1\sigma_2,m_1m_2} = \sum_{j_{12}} \begin{bmatrix} j_1 & j_2 & j_{12} \\ \sigma_1 & \sigma_2 & \sigma_1+\sigma_2 \end{bmatrix}_q \left[M^{(j_{12})}(x)\right]_{\sigma_1+\sigma_2,m_1+m_2} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_1+m_2 \end{bmatrix}_q$$

Using the interpretation of M as a 3-j symbol and the defining relation of 6-j symbols, we can write

$$\begin{aligned}&\left[M^{(j_{12})}(x)\right]_{\sigma_1+\sigma_2,m_1+m_2} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_1+m_2 \end{bmatrix}_q \quad = \\ &\sum_{\sigma_1'\sigma_2'} \frac{\mathcal{N}_\psi^{(j_{12})}(x,\sigma_1+\sigma_2)}{\mathcal{N}_\psi^{(j_1)}(xq^{2\sigma_2'},\sigma_1')\mathcal{N}_\psi^{(j_2)}(x,\sigma_2')} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j(x) & j(x)+\sigma_1+\sigma_2 & j(x)+\sigma_2' \end{Bmatrix}_q M^{(j_1)}_{\sigma_1',m_1}(xq^{2\sigma_2'})M^{(j_2)}_{\sigma_2',m_2}(x)\end{aligned}$$

Hence

$$\begin{aligned}&[F^{j_1 j_2}(x)]_{\sigma_1\sigma_2,\sigma_1'\sigma_2'} \quad = \\ &\sum_{j_{12}} \frac{\mathcal{N}_\psi^{(j_{12})}(x,\sigma_1+\sigma_2)}{\mathcal{N}_\psi^{(j_1)}(xq^{2\sigma_2'},\sigma_1')\mathcal{N}_\psi^{(j_2)}(x,\sigma_2')} \begin{bmatrix} j_1 & j_2 & j_{12} \\ \sigma_1 & \sigma_2 & \sigma_1+\sigma_2 \end{bmatrix}_q \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j(x) & j(x)+\sigma_1'+\sigma_2' & j(x)+\sigma_2' \end{Bmatrix}_q\end{aligned}$$

5 Application to the trigonometric q-deformed Lamé equation

In [3] it was shown how solutions of eq.(1) could be used to construct a set of commuting operators corresponding to q-deformations of the quantum Calogero-Moser Hamiltonians. In the $\mathcal{U}_q(sl_2)$ case, there is only one such operator once we separate the center of mass motion.

According to the general prescription [3], we start from a Lax matrix satisfying

$$R_{12}(xq^{-\frac{1}{2}H_3})L_{13}(x)L_{23}(x) = L_{23}(x)L_{13}(x)R_{12}(xq^{\frac{1}{2}H_3}), \tag{19}$$

with a subscript 3 denoting the quantum space. The first Hamiltonian is the restriction of $\mathrm{Tr}_1(L_{13}(x))$ to the subspace of zero-weight vectors.

In the representation $\rho = \rho^{(1/2)} \otimes \rho^{(1/2)}$ of $\mathcal{U}_q(sl_2) \otimes \mathcal{U}_q(sl_2)$, the following extra property is true:

$$\rho\left([(H_1 + H_2)\partial_x, R_{12}(x)]\right) = 0.$$

This condition allows to recast eq.(1) in the form (19) with a Lax operator $L(x)$ obtained by dressing $R(x)$ with suitable shift operators:

$$L_{13}(x) = q^{-(H_1+\frac{1}{2}H_3)p}\ R_{13}(x)\ q^{\frac{1}{2}H_3 p}, \quad \text{with} \quad p = x\frac{\partial}{\partial x}.$$

In the representation $\rho_j = \rho^{(1/2)} \otimes \rho^{(j)}$ of $\mathcal{U}_q(sl_2) \otimes \mathcal{U}_q(sl_2)$,

$$\rho_j(L_{13}(x)) = \begin{pmatrix} q^{-p}q^{\frac{1}{2}H} & -q^{-\frac{1}{2}}x^{-1}f(xq^{\frac{1}{2}H})q^{-\frac{1}{2}H}E_- \\ q^{-\frac{1}{2}}xf(xq^{-\frac{1}{2}H+1})q^{\frac{1}{2}H}E_+ & q^p q^{-\frac{1}{2}H}\left[1 - f(xq^{-\frac{1}{2}H})f(xq^{\frac{1}{2}H-1})E_+E_-\right] \end{pmatrix} \tag{20}$$

with $f(x) = (q - q^{-1})/(x - x^{-1})$.

Taking the trace on the first space we get

$$\mathrm{Tr}_1(L_{13}(x)) = q^{-p}q^{\frac{1}{2}H} + q^p q^{-\frac{1}{2}H}\left[1 - f(xq^{-\frac{1}{2}H})f(xq^{\frac{1}{2}H-1})E_+E_-\right].$$

We still have to restrict this operator to the space of zero-weight vectors. In the spin j, representation, when j is integer, this subspace is one-dimensional and the resulting Hamiltonian is scalar. Using $E_+E_-|j,0\rangle = [j][j+1]|j,0\rangle$, we get

$$H_j = q^{-p} + q^p\left(1 - \frac{(q-q^{-1})^2[j][j+1]}{(x-x^{-1})(q^{-1}x - qx^{-1})}\right).$$

At the first non-trivial order of H_j in the limit $q \to 1$, we recover the Calogero-Moser Hamiltonian $-\partial_z^2 + j(j+1)/\sinh^2(z)$, with $x = \exp(z)$. Notice that the coupling constant is related to the spin of the representation.

Alternatively, introducing the function

$$c_j(x) = \frac{(q^j x - q^{-j}x^{-1})(q^{-j-1}x - q^{j+1}x^{-1})}{(x-x^{-1})(q^{-1}x - qx^{-1})},$$

the Hamiltonian H_j is given by

$$H_j = q^{-p} + q^p c_j(x).$$

The eigenfunctions Ψ of H_j are the solutions of the following trigonometric q-deformed Lamé equation:

$$\Psi(q^{-1}x) + c_j(qx)\Psi(qx) = E\ \Psi(x).$$

An elliptic version of this equation already appeared in a different context in [14].

Integrability of the system manifests itself in the fact that we can easily solve this equation by using, for instance, the following recursive procedure. For $j = 0$ the Hamiltonian $H_0 = q^p + q^{-p}$ is free; its eigenfunctions are plane waves $\Psi_0(x,k) = x^k$ with the corresponding energy $E(k) = q^k + q^{-k}$. Let us now introduce the following "shift" operator

$$D_j = q^{-p} - q^p \frac{(q^{-j}x - q^j x^{-1})(q^{-j-1}x - q^{j+1}x^{-1})}{(x - x^{-1})(q^{-1}x - qx^{-1})}$$

which satisfies

$$H_j D_j = D_j H_{j-1},$$

The eigenfunction $\Psi_j(x,k)$ of H_j with energy $E(k) = q^k + q^{-k}$, are obtained by the successive action of the "shift" operator:

$$\Psi_j(x,k) = D_j\ \Psi_{j-1}(x,k).$$

Since the energy is even in k, we can start the recursion with $\Psi_0(x,k) = x^k - x^{-k}$. Then we get

$$\Psi_j(x,k) \;=\; \sum_{n=0}^{j} (-1)^n \begin{bmatrix} j \\ n \end{bmatrix}_q \frac{\prod_{r=1}^{n}(q^{r-j-1}x - q^{-r+j+1}x^{-1})}{\prod_{r=1}^{n}(q^r x - q^{-r}x^{-1})} (q^{k(2n-j)}x^k - q^{-k(2n-j)}x^{-k})$$

This wave function has the interesting properties that the residues at the poles $x = \pm q^{-r}$ for $1 \le r \le j$ all vanish, and moreover, one has $\Psi_j(x,k) = 0$ for $k = -j, -j+1, \cdots, j$. This is an analogue of the generalized exclusion principle present in the Calogero-Sutherland model [15].

6 Appendix

We give an idea of the proof of eq.(14). We adopt here a naive point of vue. We refer to [11] for a more detailed discussion. We start from an avatar of van der Waerden formula for 3-j symbols (combine eq. 3.5 and eq. 3.10 in ref.[13]):

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix}_q = \delta_{m_1+m_2,m_3}\ \Delta(j_1,j_2,j_3)\ q^{-\frac{1}{2}(j_1+j_2-j_3)(j_1+j_2+j_3+1)+j_1 m_2 - j_2 m_1}\sqrt{[2j_3+1]}\cdot$$
$$\cdot\sqrt{[j_1+m_1]![j_1-m_1]![j_2+m_2]![j_2-m_2]![j_3+m_3]![j_3-m_3]!}\cdot$$
$$\cdot\sum_p \frac{(-1)^p q^{p(j_1+j_2+j_3+1)}}{[p]![j_1+j_2-j_3-p]![j_2-m_2-p]![j_1+m_1-p]![j_3-j_1+m_2+p]![j_3-j_2-m_1+p]!}$$

where

$$\Delta(j_1,j_2,j_3) \;=\; (-1)^{j_1+j_2-j_3}\sqrt{\frac{[-j_1+j_2+j_3]![j_1-j_2+j_3]![j_1+j_2-j_3]!}{[j_1+j_2+j_3+1]!}}$$

We take a limit $m_2 \to \infty$ such that

$$\lim_{m_2\to\infty} q^{m_2} = 0, \quad \lim_{m_2\to\infty} q^{-m_2} = \infty$$

Then, one has

$$\frac{[\alpha \pm m_2]!}{[\beta \pm m_2]!} \sim (\mp)^{\alpha-\beta} \frac{q^{\mp\frac{1}{2}(\alpha-\beta)(\alpha+\beta+1)}}{(q-q^{-1})^{\alpha-\beta}} q^{-(\alpha-\beta)m_2}$$

To perform the limit, we write the terms containing m_2 in the following form

$$\sqrt{\frac{[j_2+m_2]!}{[j_3-j_1+m_2]!} \cdot \frac{[j_3+m_1+m_2]!}{[j_3-j_1+m_2]!} \cdot \frac{[j_2-m_2]!}{[j_2-m_2]!} \cdot \frac{[j_3-m_1-m_2]!}{[j_2-m_2]!}} \sim$$
$$(-1)^{j_1+\frac{1}{2}(j_2-j_3+m_1)} \frac{q^{\frac{1}{2}(-2j_3m_1-m_1-2j_1(j_3+1)+j_1(j_1+1)+j_3(j_3+1)-j_2(j_2+1))}}{(q-q^{-1})^{j_1}} q^{-j_1m_2}$$

and

$$\lim_{m_2\to\infty} \frac{[j_2-m_2]!\ [j_3-j_1+m_2]!}{[j_2-m_2-p]![j_3-j_1+m_2+p]!} = (-1)^p q^{p(j_2+j_3-j_1+1)}$$

This decomposition is to ensure that we get the above important sign $(-1)^p$ correctly. Hence

$$\begin{aligned}\lim_{m_2\to\infty} \begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_1+m_2 \end{bmatrix}_q &= \Delta(j_1,j_2,j_3) \frac{\sqrt{[2j_3+1]}\sqrt{[j_1+m_1]![j_1-m_1]!}}{(q-q^{-1})^{j_1}} \cdot \\ &\cdot (-1)^{j_1+\frac{1}{2}(j_2-j_3+m_1)} q^{-j_2(j_2+1)+j_3(j_3+1)-j_1(j_3+j_2+1)} q^{-\frac{1}{2}m_1} \cdot \\ &\cdot q^{-(j_2+j_3)m_1} \sum_p \frac{q^{2p(j_2+j_3+1)}}{[p]![j_1+j_2-j_3-p]![j_1+m_1-p]![j_3-j_2-m_1+p]!}\end{aligned}$$

Comparing with eq.(13) we get eq.(14) with $j_2 = j(x)$ and $j_3 = j(x)+\sigma_1$ where $j(x)$ is given by eq.(15). Moreover we find

$$\begin{aligned}\mathcal{N}_\xi^{(j_1)}(m_1) &= (-1)^{-\frac{1}{2}m_1} q^{\frac{1}{2}m_1} \\ \mathcal{N}_\psi^{(j_1)}(x,\sigma_1) &= (-1)^{-j_1+\frac{1}{2}(m_1-\sigma_1)} \frac{\sqrt{[j_1+\sigma_1]![j_1-\sigma_1]!}}{\prod_{r=1}^{j_1+\sigma_1}(1-x^2q^{2r})} \frac{(q-q^{-1})^{j_1} x^{j_1} q^{j_1\sigma_1}}{\Delta(j_1,j(x),j(x)+\sigma_1)\sqrt{[2j(x)+1+\sigma_1]}}\end{aligned}$$

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